

AD 740567

ARL 71-0319



Aerospace Research Laboratories

ON A CLASS OF SIMULTANEOUS RANK ORDER TESTS IN MANOCOVA

P. K. SEN

*UNIVERSITY OF NORTH CAROLINA
CHAPEL HILL, NORTH CAROLINA*

P. R. KRISHNAIAH

APPLIED MATHEMATICS RESEARCH LABORATORY

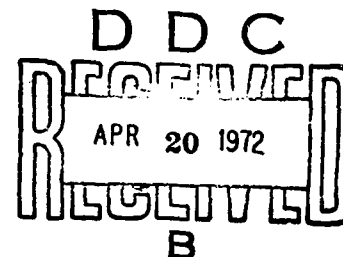
PROJECT NO. 7071

Approved for public release; distribution unlimited.

Reproduced by
NATIONAL TECHNICAL
INFORMATION SERVICE
Springfield, Va. 22151

AIR FORCE SYSTEMS COMMAND

United States Air Force



NOTICES

When Government drawings, specifications, or other data are used for any purpose other than in connection with a definitely related Government procurement operation, the United States Government thereby incurs no responsibility nor any obligation whatsoever; and the fact that the Government may have formulated, furnished, or in any way supplied the said drawings, specifications, or other data, is not to be regarded by implication or otherwise as in any manner licensing the holder or any other person or corporation, or conveying any rights or permission to manufacture, use, or sell any patented invention that may in any way be related thereto.

Agencies of the Department of Defense, qualified contractors and other government agencies may obtain copies from the

Defense Documentation Center
Cameron Station
Alexandria, Virginia 22314


This document has been released to the

CLEARINGHOUSE
U. S. Department of Commerce
Springfield, Virginia 22151

for the sale to the public.

Copies of ARL Technical Documentary Reports should not be returned to Aerospace Research Laboratories unless return is required by security considerations, contractual obligations or notices on a specified document.

AIR FORCE: 8-3-72/500

SEARCHED FOR		WHITE SECTION <input checked="" type="checkbox"/>
ASTI	DOC	REF SECTION <input type="checkbox"/>
ANNOUNCED		<input type="checkbox"/>
IDENTIFICATION		<input type="checkbox"/>
BY	DISTRIBUTION/AVAILABILITY CODES	
DIS.	AVAIL. and/or SPECIAL	
		

Unclassified

Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) Aerospace Research Laboratories Applied Mathematics Research Laboratory Wright-Patterson AFB, Ohio 45433		2a. REPORT SECURITY CLASSIFICATION Unclassified	
3. REPORT TITLE On a Class of Simultaneous Rank Order Tests in MANOCOVA		2b. GROUP	
4. DESCRIPTIVE NOTES (Type of report and inclusive dates) Scientific Interim			
5. AUTHOR(S) (First name, middle initial, last name) P. K. Sen and P. R. Krishnaiah			
6. REPORT DATE December 1971	7a. TOTAL NO. OF PAGES 17	7b. NO. OF REFS 11	
8a. PROJECT NO. In-House Research		9a. ORIGINATOR'S REPORT NUMBER(S)	
b. PROJECT NO. 7071-00-12			
c. DoD Element 61102 F		9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)	
d. DoD Subelement 681304		ARL 71-0319	
10. DISTRIBUTION STATEMENT Approved for public release; distribution unlimited			
11. SUPPLEMENTARY NOTES TECH OTHER		12. SPONSORING MILITARY ACTIVITY Aerospace Research Laboratories (LB) Wright-Patterson AFB, Ohio 45433	
13. ABSTRACT For the one-criterion multivariate analysis of covariance (MANOCOVA) model, the rank order tests for the overall hypothesis of no treatment effect considered by Quade (1967), Puri and Sen (1969) and Sen and Puri (1970) are extended here to some simultaneous tests for various component hypotheses. The theory is based on an extension of rank order estimates of contrasts in multivariate analysis of variance (MANOVA) developed by Puri and Sen (1968) to the MANOCOVA problem, and is formulated in the set up of Gabriel and Sen (1968) and Krishnaiah (1964,1969).			

DD FORM 1 NOV 65 1473

Unclassified

Security Classification

ARL 71-0319

**ON A CLASS OF SIMULTANEOUS
RANK ORDER TESTS IN MANOCOVA**

P. K. SEN

*UNIVERSITY OF NORTH CAROLINA
CHAPEL HILL, NORTH CAROLINA*

P. R. KRISHNAIAH

APPLIED MATHEMATICS RESEARCH LABORATORY

DECEMBER 1971

PROJECT NO. 7071

Approved for public release; distribution unlimited.

AEROSPACE RESEARCH LABORATORIES
AIR FORCE SYSTEMS COMMAND
UNITED STATES AIR FORCE
WRIGHT-PATTERSON AIR FORCE BASE, OHIO

FOREWORD

This report was prepared for the Applied Mathematics Research Laboratories by P. K. Sen and P. R. Krishnaiah under Project 7071, "Research in Applied Mathematics". Part of the work of P. K. Sen was performed at the Aerospace Research Laboratories under Contract AF 33(615)-2272 of the above laboratories with the University of Cincinnati. The work of P. K. Sen, in the final stage, was performed under Contract F 33615-71-C-1927 of the Aerospace Research Laboratories with the University of North Carolina.

ABSTRACT

For the one-criterion multivariate analysis of covariance (MANOCOVA) model, the rank order tests for the overall hypothesis of no treatment effect considered by Quade (1967), Puri and Sen(1969) and Sen and Puri(1970) are extended here to some simultaneous tests for various component hypotheses. The theory is based on an extension of rank order estimates of contrasts in multivariate analysis of variance (MANOVA) developed by Puri and Sen(1968) to the MANOCOVA problem, and is formulated in the set up of Gabriel and Sen (1968) and Krishnaiah(1964,1969).

TABLES OF CONTENTS

Section	Page
1. Introduction	1
2. Covariate Adjusted Rank Order Estimates of Contrasts in MANOCOVA	1
3. Simultaneous Test Procedures	7
4. ARE Results	10
References	12

1. INTRODUCTION

Let $Z_{\alpha}^{(k)} = (Y_{\alpha}^{(k)}, X_{\alpha}^{(k)})' = (Y_{1\alpha}^{(k)}, \dots, Y_{p\alpha}^{(k)}, X_{1\alpha}^{(k)}, \dots, X_{q\alpha}^{(k)})'$, $\alpha = 1, \dots, n_k$ be n_k independent and identically distributed random vectors (iidrv) with a continuous cumulative distribution function (cdf) $\Pi_k(z)$, $z \in R^{p+q}$, the $(p+q)$ -dimensional Euclidean space, where $p \geq 1, q \geq 1$, and $k = 1, \dots, c (\geq 2)$; all these $N (= \sum_{k=1}^c n_k)$ stochastic vectors are assumed to be independent. The q -variate marginal cdf of $X_{\alpha}^{(k)}$ is denoted by $G_k(x)$, $x \in R^q$, and the p -variate conditional cdf of $Y_{\alpha}^{(k)}$, given $X_{\alpha}^{(k)} = x$, is denoted by $F_k(y | x)$, $y \in R^p$, $x \in R^q$, $k=1, \dots, c$. Our basic model is the following :

$$G_k(x) = G(x) \quad \text{and} \quad F_k(y | x) = F(y - \tau_k | x), \quad \text{for } k=1, \dots, c, \quad (1.1)$$

where F and G are unknown continuous cdfs, and τ_1, \dots, τ_c are the unknown treatment effects (p -vectors). For justification of this model, we may refer to Scheffe' (1959, ch.6) and Sen and Puri(1970). Tests for the overall hypothesis

$$H_0: \tau_1 = \dots = \tau_c = 0 \quad \text{vs.} \quad \tau_k \neq 0 \quad \text{for at least one } k (1 \leq k \leq c), \quad (1.2)$$

based on suitable rank order statistics, were considered in an increasing order of generality by Quade(1967), Puri and Sen(1969), and Sen and Puri(1970). We are interested here in testing simultaneously for contrasts among τ_1, \dots, τ_c , both in the set up of Gabriel and Sen(1968) and Krishnaiah (1964, 1969). These simultaneous procedures involve the concomitant variates adjusted rank order estimates of contrasts among τ_1, \dots, τ_c , which are considered first in section 2. Section 3 deals with the proposed simultaneous tests, and their asymptotic relative efficiency (ARE) results are presented in the last section.

2. COVARIATE ADJUSTED RANK ORDER ESTIMATES OF CONTRASTS IN MANOCOVA

The statistics considered by Quade(1967), Puri and Sen(1969) and Sen and Puri (1970) , being based on the combined sample ranking, are not suitable for our simultaneous inference procedures to be considered in the next section. For this reason, we extend the results of Puri and Sen(1968) on rank order estimates in MANOVA to the MANOCOVA problem, which provides the access to our proposed procedures.

Consider the pair (k, ℓ) of samples, and let $n_{k\ell} = n_k + n_\ell$, for $1 \leq k < \ell \leq c$. Let $R_{i,\alpha}^{(k,\ell)}$ be the rank of $Y_{i\alpha}^{(k)}$ among the $n_{k\ell}$ observations $Y_{i1}^{(k)}, \dots, Y_{in_k}^{(k)}, Y_{i1}^{(\ell)}, \dots, Y_{in_\ell}^{(\ell)}$, for $\alpha = 1, \dots, n_k$, $1 \leq i \leq p$, and let $S_{i,\alpha}^{(k,\ell)}$ be the rank of $X_{i\alpha}^{(k)}$ among the $n_{k\ell}$ observations $X_{i1}^{(k)}, \dots, X_{in_k}^{(k)}, X_{i1}^{(\ell)}, \dots, X_{in_\ell}^{(\ell)}$, $\alpha = 1, \dots, n_k$, $i=1, \dots, q$, for $1 \leq k < \ell \leq c$. Consider now $(p+q)$ sets of rank scores

$$J_{n_{k\ell}}^{(i)}(\alpha/(n_{k\ell} + 1)), \alpha = 1, \dots, n_{k\ell}, i=1, \dots, p \text{ (primary variate scores)} \quad (2.1)$$

$$J_{n_{k\ell}}^{*(j)}(\alpha/(n_{k\ell} + 1)) \alpha = 1, \dots, n_{k\ell}, j=1, \dots, q \text{ (covariate scores)}, \quad (2.2)$$

defined for every (k, ℓ) : $1 \leq k < \ell \leq c$, in the same fashion as in Puri and Sen (1971, section 3.6). Define then the statistics

$$T_{k\ell}^{(i)} = n_k^{-1} \sum_{\alpha=1}^{n_k} J_{n_{k\ell}}^{(i)}(R_{i,\alpha}^{(k,\ell)}/(n_{k\ell} + 1)), i=1, \dots, p; 1 \leq k < \ell \leq c, \quad (2.3)$$

$$T_{k\ell}^{*(j)} = n_k^{-1} \sum_{\alpha=1}^{n_k} J_{n_{k\ell}}^{*(j)}(S_{j,\alpha}^{(k,\ell)}/(n_{k\ell} + 1)), j=1, \dots, q; 1 \leq k < \ell \leq c. \quad (2.4)$$

If instead of $Y_{i\alpha}^{(k)}$ and $Y_{i\alpha}^{(\ell)}$, one works with $Y_{i\alpha}^{(k)} - a$, $\alpha=1, \dots, n_k$ and $Y_{i\alpha}^{(\ell)}$, $\alpha = 1, \dots, n_\ell$, the corresponding rank order statistic, defined by (2.3) is denoted by $T_{k\ell}^{(i)}(a)$, $1 \leq k < \ell \leq c$, $i=1, \dots, p$, and $-\infty < a < \infty$. We now assume that $J_{n_{k\ell}}^{(i)}(u)$ is \uparrow in u : $0 < u < 1$, so that

$$T_{k\ell}^{(i)}(a) \text{ is } \downarrow \text{ in } a : -\infty < a < \infty. \quad (2.5)$$

Next, we rank the observations coordinatewise within each sample separately. Let $R_{i,\alpha}^{*(k)}$ (or $S_{i,\alpha}^{*(k)}$) be the rank of $Y_{i\alpha}^{(k)}$ among $Y_{i1}^{(k)}, \dots, Y_{in_k}^{(k)}$ (or $X_{i\alpha}^{(k)}$ among $X_{i1}^{(k)}, \dots, X_{in_k}^{(k)}$), for $\alpha = 1, \dots, n_k$, $i=1, \dots, p$ (or $i=1, \dots, q$), and $k=1, \dots, c$. Define $J_{n_k}^{(i)}$, $i=1, \dots, p$, and $J_{n_k}^{*(j)}$, $j=1, \dots, q$ as in (2.1) and (2.2) with the $n_{k\ell}$ being replaced by n_k , and let

$$v_{ij}^{(k)} = n_k^{-1} \sum_{\alpha=1}^{n_k} J_{n_k}^{(i)}(R_{i,\alpha}^{*(k)}/(n_k + 1)) J_{n_k}^{*(j)}(R_{j,\alpha}^{*(k)}/(n_k + 1)) - \bar{J}_{n_k}^{(i)} \bar{J}_{n_k}^{*(j)}, \quad (2.6)$$

for $i, j = 1, \dots, p$, where

$$\bar{J}_{n_k}^{(i)} = n_k^{-1} \sum_{\alpha=1}^{n_k} J_{n_k}^{(i)}(\alpha/(n_k + 1)), i = 1, \dots, p. \quad (2.7)$$

Similarly, define $v_{ij}^{(k)}$ for $p+1 \leq i, j \leq p+q$ by replacing $J_{n_k}^{(i)}$ and $R_{i,\alpha}^{*(k)}$ by $J_{n_k}^{*(i)}$ and $S_{i,\alpha}^{*(k)}$ respectively (and also for j) in the definitions in (2.6) and (2.7). Finally, keeping $J_{n_k}^{(i)}$ and $R_{i,\alpha}^{*(k)}$ as they are while replacing $J_{n_k}^{(j)}$ and $R_{j,\alpha}^{*(k)}$ by $J_{n_k}^{*(j)}$ and $S_{j,\alpha}^{*(k)}$, respectively, in (2.6) and (2.7), we define in the same way $v_{ij}^{(k)}$ for $i=1, \dots, p; j=p+1, \dots, p+q$. Let then

$$\bar{v}^{(k)} = ((v_{ij}^{(k)}))_{i,j=1, \dots, p+q}, \quad k=1, \dots, c, \quad (2.8)$$

where $v_{ji}^{(k)} = v_{ij}^{(k)}$ for all i, j . Further, let

$$\bar{v}_N = \sum_{k=1}^c (n_k/N) \bar{v}^{(k)} = \begin{pmatrix} \bar{v}_{N,11} & \bar{v}_{N,12} \\ \bar{v}_{N,21} & \bar{v}_{N,22} \end{pmatrix}, \quad (2.9)$$

where $\bar{v}_{N,11}, \bar{v}_{N,12} = \bar{v}_{N,21}'$ and $\bar{v}_{N,22}$ are respectively of the order $p \times p, p \times q$ and $q \times q$. Finally, let

$$\bar{v}_N^* = \bar{v}_{N,11} - \bar{v}_{N,12} \bar{v}_{N,22}^- \bar{v}_{N,21}, \quad (2.10)$$

where $\bar{v}_{N,22}^-$ is a generalized inverse of $\bar{v}_{N,22}$. Define then $\bar{J}_{n_{k\ell}}^{(i)}, i=1, \dots, p$, and $\bar{J}_{n_{k\ell}}^{*(j)}, j=1, \dots, q$, for $1 \leq k < \ell \leq c$ as in (2.7) with n_k being replaced by $n_{k\ell}$, and let

$$T_{k\ell}(a) = (T_{k\ell}^{(1)}(a_1), \dots, T_{k\ell}^{(p)}(a_p))', \quad 1 \leq k < \ell \leq c; \quad (2.11)$$

$$T_{k\ell}^* = (T_k^{*(1)}, \dots, T_k^{*(q)})', \quad 1 \leq k < \ell \leq c; \quad (2.12)$$

$$\bar{J}_{k\ell} = (\bar{J}_{k\ell}^{(1)}, \dots, \bar{J}_{k\ell}^{(p)})', \quad \bar{J}_k^* = (\bar{J}_{k\ell}^{*(1)}, \dots, \bar{J}_{k\ell}^{*(q)})', \quad 1 \leq k < \ell \leq c; \quad (2.13)$$

$$\begin{aligned} T_{k\ell}^o(a) &= (T_{k\ell}(a) - \bar{J}_{n_{k\ell}}) - \bar{v}_{N,12} \bar{v}_{N,22}^- (\bar{J}_k^* - \bar{J}_{k\ell}^*) \\ &= (T_{k\ell}^{o(1)}(a_1), \dots, T_{k\ell}^{o(p)}(a_p))', \quad \text{for } 1 \leq k < \ell \leq c. \end{aligned} \quad (2.14)$$

The statistics in (2.14) are the covariate adjusted rank order statistics as defined in Sen and Puri(1970) with the only difference that their combined sample permutation covariance matrix is replaced here by \bar{v}_N , the average within sample rank order covariance matrix. This replacement is made with the primary objective of making the adjustments for the covariates in (2.14) unaffected by τ_1, \dots, τ_c ; \bar{v}_N is

translation invariant, so that the unknown τ_1, \dots, τ_c have no effect on it, while the combined sample permutation covariance matrix rests on the assumption that

$\tau_1 = \dots = \tau_c$, which is not necessarily true.

Now, precisely by the same alignment logic as in Puri and Sen(1968), we define

$$\hat{\Delta}_{k\ell,1}^{(i)} = \inf\{a: T_k^{o(i)}(a) < 0\}, \quad \hat{\Delta}_{k\ell,2}^{(i)} = \sup\{a: T_k^{o(i)}(a) > 0\}, \quad (2.15)$$

for $1 \leq k < \ell \leq c$, $i=1, \dots, p$. Then, our proposed estimator of $\Delta_{k\ell} = \tau_k - \tau_\ell$ is

$$\hat{\Delta}_{k\ell} = (\hat{\Delta}_{k\ell}^{(1)}, \dots, \hat{\Delta}_{k\ell}^{(p)})'; \quad \hat{\Delta}_{k\ell} = \frac{1}{2} (\hat{\Delta}_{k\ell,1}^{(i)} + \hat{\Delta}_{k\ell,2}^{(i)}), \quad 1 \leq i \leq p. \quad (2.16)$$

Conventionally, we let $\Delta_{kk} = \hat{\Delta}_{kk} = 0$, $k=1, \dots, c$ and $\Delta_{k\ell} = -\Delta_{\ell k}$ for $1 \leq k < \ell \leq c$.

Then, as in Puri and Sen (1968; 1971, Ch. 6), we define the compatible estimators

$$\hat{\Delta}_{k\ell}^* = \hat{\Delta}_{k\ell} - \hat{\Delta}_{\ell\ell}; \quad \hat{\Delta}_{k\ell}^* = c^{-1} \sum_{\ell=1}^c \hat{\Delta}_{k\ell}, \quad 1 \leq k < \ell \leq c. \quad (2.17)$$

Let $H_{(i)}(x)$ (and $G_{(j)}(x)$) be the marginal cdf of $Y_{i\alpha}^{(k)} - \tau_k^{(i)}$ (and $X_{i\alpha}^{(k)}$), $i=1, \dots, p$ (and $j=1, \dots, q$), and let $H_{(i,i')}(x,y)$, $H_{(i,j)}^{(2)}(x,y)$ and $G_{(j,j')}(x,y)$ be respectively the joint cdf of $(Y_{i\alpha}^{(k)} - \tau_k^{(i)}, Y_{i'\alpha}^{(k)} - \tau_k^{(i')})$, $(Y_{i\alpha}^{(k)} - \tau_k^{(i)}, X_{j\alpha}^{(k)})$ and $(X_{j\alpha}^{(k)}, X_{j'\alpha}^{(k)})$, for $i(\neq i') = 1, \dots, p$ and $j(\neq j') = 1, \dots, q$. Define then

$$J^{(i)}(u) = \lim_{n \rightarrow \infty} J_n^{(i)}(u), \quad J^{*(j)}(u) = \lim_{n \rightarrow \infty} J_n^{*(j)}(u), \quad 0 < u < 1, \quad (2.18)$$

for $i=1, \dots, p$, $j=1, \dots, q$, and denote by

$$\mu_i = \int_0^1 J^{(i)}(u) du, \quad i=1, \dots, p; \quad \mu_j^* = \int_0^1 J^{*(j)}(u) du, \quad j=1, \dots, q; \quad (2.19)$$

$$\begin{aligned} v_{ij,11} &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} J^{(i)}(H_{(i)}(x)) J^{(j)}(H_{(j)}(y)) dH_{(i,j)}^{(1)}(x,y) \\ &\quad - \mu_i \mu_j, \quad i, j = 1, \dots, p; \end{aligned} \quad (2.20)$$

$$\begin{aligned} v_{ij,12} &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} J^{(i)}(H_{(i)}(x)) J^{*(j)}(G_{(j)}(y)) dH_{(i,j)}^{(2)}(x,y) \\ &\quad - \mu_i \mu_j^*, \quad i=1, \dots, p; \quad j=1, \dots, q; \end{aligned} \quad (2.21)$$

$$\begin{aligned} v_{ij,22} &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} J^{*(i)}(G_{(i)}(x)) J^{*(j)}(G_{(j)}(y)) dG_{(i,j)}(x,y) \\ &\quad - \mu_i^* \mu_j^*, \quad i, j = 1, \dots, q; \end{aligned} \quad (2.22)$$

$$\begin{aligned} \tilde{v} &= \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix} \quad v_{11} = ((v_{ij,11})), \quad v_{22} = ((v_{ij,22})), \end{aligned} \quad (2.23)$$

and $\tilde{v}_{12} = \tilde{v}_{21}' = ((v_{ij,12}))$. Further, let

$$\tilde{v}^* = \tilde{v}_{11} - \tilde{v}_{12} \tilde{v}_{22}^{-1} \tilde{v}_{21} = ((v_{ij}^*))_{i,j=1,\dots,p}; \quad (2.24)$$

$$\tilde{\Gamma}^* = ((\gamma_{ij}^*)) = ((v_{ij}/B_i B_j))_{i,j=1,\dots,p}, \quad (2.25)$$

where

$$B_i = \int_{-\infty}^{\infty} (d/dx) J^{(i)}(H_{(i)}(x)) dH_{(i)}(x), \quad i = 1, \dots, p. \quad (2.26)$$

Finally, consider a contrast

$$\phi(\underline{\ell}) = \ell_1 \tau_1 + \dots + \ell_c \tau_c, \quad \underline{\ell} = (\ell_1, \dots, \ell_c)', \quad \underline{1} = (1, \dots, 1)', \quad (2.27)$$

for which we have the compatible estimator

$$\hat{\phi}^*(\underline{\ell}) = \ell_1 \hat{\Delta}_1 + \dots + \ell_c \hat{\Delta}_c, \quad (2.28)$$

where the $\hat{\Delta}_k$ are defined by (2.17). For any $\underline{\ell} \perp \underline{1}$, (2.28) provides a robust, translation invariant and concomitant variate adjusted estimator.

For the study of the asymptotic properties of $\hat{\phi}^*(\underline{\ell})$, we assume that as $N \rightarrow \infty$,

$$n_k/N \rightarrow \lambda_k : 0 < \lambda_k < 1, \text{ for } k = 1, \dots, c. \quad (2.29)$$

Then, by the same technique as in theorems 2.1 and 2.2 of Puri and Sen(1968) (with direct adaptations from theorem 5.1 of Puri and Sen(1966), yielding the joint asymptotic normality of the covariate adjusted rank order statistics in (2.14)), we obtain that

$$\mathcal{L}(N^{1/2} [\hat{\phi}^*(\underline{\ell}) - \phi(\underline{\ell})]) \rightarrow \mathcal{N}(0, (\sum_{k=1}^c \ell_k^2 / \lambda_k) \tilde{\Gamma}^*). \quad (2.30)$$

Similarly, if we have $r(\geq 1)$ linearly independent contrasts

$$\phi(\underline{\ell}_s) = \sum_{k=1}^c \ell_{sk} \tau_k, \quad \underline{\ell}_s \perp \underline{1}, \quad s = 1, \dots, r, \quad (2.31)$$

and we define $\underline{C} = ((c_{ss'}))_{s,s'=1,\dots,r}$ by $c_{ss'} = \sum_{k=1}^c \ell_{sk} \ell_{s'k} / \lambda_k$, $s, s' = 1, \dots, r$, we have

$$\mathcal{L}(N^{1/2} [\hat{\phi}^*(\underline{\ell}_s) - \phi(\underline{\ell}_s)], 1 \leq s \leq r) \rightarrow \mathcal{N}(0, \underline{C} \otimes \tilde{\Gamma}^*), \quad (2.32)$$

where \otimes stands for the Kronecker product of two matrices.

We shall make use of (2.30) and (2.32) for providing suitable confidence intervals to $\phi(\underline{\ell})$, $\underline{\ell} \perp \underline{1}$. For this, we require to estimate the unknown $\tilde{\Gamma}^*$. Now, on

defining t_α as the upper 100 α % point of the standard normal distribution, and letting

$$\hat{\Delta}_{k\ell,U}^{(i)} = \sup\{a: T_{k\ell}^{o(i)}(a) > -t_{\alpha/2} [\bar{v}_{N,ii}^* (n_{k\ell}/n_k n_\ell)]^{1/2}\}, \quad (2.33)$$

$$\hat{\Delta}_{k\ell,L}^{(i)} = \inf\{a: T_k^{o(i)}(a) < t_{\alpha/2} [\bar{v}_{N,ii}^* (n_{k\ell}/n_k n_\ell)]^{1/2}\}, \quad (2.34)$$

and following the same method of proof as in theorem 1 of Sen(1966), it follows that

$$\hat{B}_{k\ell}^{(i)} = (2t_{\alpha/2} (\bar{v}_{N,ii}^*)^{1/2}) / [(n_{k\ell}/n_k n_\ell)^{1/2} (\hat{\Delta}_{k\ell,U}^{(i)} - \hat{\Delta}_{k\ell,L}^{(i)})], \quad 1 \leq k < \ell \leq c \quad (2.35)$$

are all translation invariant consistent estimators of B_i , defined by (2.26), for $i=1, \dots, p$. Hence,

$$\hat{B}^{(i)} = \binom{c}{2}^{-1} \sum_{1 \leq k < \ell \leq c} \hat{B}_{k\ell}^{(i)}, \quad i = 1, \dots, p \quad (2.36)$$

are translation invariant robust and consistent estimators of B_1, \dots, B_p , respectively.

Also, by theorem 4.2 of Puri and Sen(1966) [see also theorem 3.2 of Ghosh and Sen(1971)] and some standard manipulations, it follows that whatever be τ_1, \dots, τ_c ,

$$\bar{v}_N \xrightarrow{P} \underline{v} \quad \text{as } N \rightarrow \infty. \quad (2.37)$$

Thus, from (2.10), (2.24), (2.25), (2.36) and (2.37), we obtain on defining

$$\hat{\Gamma}_N^* = ((\hat{\gamma}_{ij}^*)) ; \quad \hat{\gamma}_{ij}^* = \bar{v}_{N,ij}^* / [\hat{B}^{(i)} \hat{B}^{(j)}], \quad i, j = 1, \dots, p. \quad (2.38)$$

that

$$\hat{\Gamma}_N^* \xrightarrow{P} \underline{\Gamma}^* \quad \text{as } N \rightarrow \infty. \quad (2.39)$$

Now, the space of all possible contrasts $\phi(\underline{\ell}), \underline{\ell} \perp \underline{1}$, is spanned by a set of $c-1$ linearly independent contrasts. Hence, using (2.31) and (2.32) with $r = c-1$, and (2.39), we obtain by Roy-Bose method(1953) that

$$\begin{aligned} \lim_{N \rightarrow \infty} P\{N^{1/2} [t'(\underline{\phi}^*(\underline{\ell}) - \underline{\phi}(\underline{\ell}))][(\underline{t}' \hat{\Gamma}_N^* \underline{t})(\sum_{k=1}^c \ell_k^2 / \lambda_k)]^{-1/2} \\ \leq \chi_{p(c-1), \alpha} \text{ for every } \underline{\ell} \perp \underline{1}, \underline{t} \neq \underline{0}\} = 1 - \alpha, \end{aligned} \quad (2.40)$$

which provide a simultaneous confidence region to the set of all possible contrasts where $\chi_{p(c-1), \alpha}^2$ is the upper 100 % point of the chi-square distribution with $p(c-1)$ degrees of freedom. ARE results will be considered in section 4.

3. SIMULTANEOUS TEST PROCEDURES

Along the lines of Krishnaiah (1964, 1969), we consider here some simultaneous test procedures (STP) for contrasts in MANOCOVA based on the robust estimators derived in the earlier section.

Procedure I. Consider a set of $r(> 1)$ linearly independent contrasts (vectors)

$\phi(\underline{\ell}_s)$, $s = 1, \dots, r$, defined as in (2.31), and let

$$H_s: \phi(\underline{\ell}_s) = \underline{0} \quad \text{vs} \quad A_s: \phi(\underline{\ell}_s) \neq \underline{0}, \quad s = 1, \dots, r; \quad (3.1)$$

$$H = H_1 \cap \dots \cap H_r \quad \text{and} \quad A = A_1 \cup \dots \cup A_r. \quad (3.2)$$

Let us define the compatible estimators $\phi^*(\underline{\ell}_s)$, $s = 1, \dots, r$, as in (2.28), and let

$$Q_{N,s} = (N/c_{ss})[\phi^*(\underline{\ell}_s)]'(\hat{\Gamma}_N^*)^{-1}[\phi^*(\underline{\ell}_s)], \quad s = 1, \dots, r, \quad (3.3)$$

where $\hat{\Gamma}_N^*$ is defined by (2.38), and c_{ss} , in between (2.31) and (2.32). By using (2.32), (2.39) and a result of Krishnomoorthy and Parthasarathy (1951), we conclude that under H in (3.2), $\underline{Q}_N = (Q_{N,1}, \dots, Q_{N,r})'$ has asymptotically a multivariate chi-square distribution. Hence, there exists a Q_α such that

$$\lim_{N \rightarrow \infty} P\{Q_{N,s} \leq Q_\alpha, \forall s = 1, \dots, r \mid H\} = 1 - \alpha. \quad (3.4)$$

Our proposed STP consists in rejecting H_s in favor of A_s only for those s for which $Q_{N,s} > Q_\alpha$; otherwise, accept H_s , $s = 1, \dots, r$. The total hypothesis H is accepted iff all the component hypotheses are accepted. The associated simultaneous confidence intervals for $t'\phi(\underline{\ell}_s)$, $t \neq 0$, are

$$|t'(\phi^*(\underline{\ell}_s) - \phi(\underline{\ell}_s))| \leq \{Q_\alpha c_{ss} (t'\hat{\Gamma}_N^* t)/N\}^{1/2}, \quad s = 1, \dots, r, \quad t \neq 0, \quad (3.5)$$

with an overall confidence coefficient, asymptotically, equal to $1 - \alpha$.

Procedure II. This is really a stepdown procedure. We denote by \underline{C}_j the principal

minor of \underline{C} comprising of the first j rows and columns, and write for $2 \leq j \leq r$,

$$\underline{C}_j = \begin{pmatrix} \underline{C}_{j-1} & \underline{c}_{j-1} \\ \underline{c}_{j-1}' & c_{jj} \end{pmatrix}, \quad c_{jj}^* = c_{jj} - \underline{c}_{j-1}' \underline{C}_{j-1}^{-1} \underline{c}_{j-1}, \quad j = 2, \dots, r, \quad \text{and let } c_{11}^* = c_{11}.$$

Then, by (2.32), on denoting by $\underline{Z}_1 = N^{1/2}[\phi^*(\underline{\ell}_1) - \phi(\underline{\ell}_1)]$, and for $j \geq 2$,

$$Z_j = N^{1/2}[(\phi^*(\xi_1) - \phi(\xi_1))', \dots, (\phi^*(\xi_{j-1}) - \phi(\xi_{j-1}))']', \quad (3.6)$$

we conclude that Z_1 has asymptotically a multinormal distribution with mean 0 and dispersion matrix $c_{11} \Gamma_N^*$, while given Z_j , the conditional distribution of $N^{1/2}[\phi^*(\xi_j) - \phi(\xi_j)]$ is asymptotically multinormal with mean vector

$$(c'_{j-1} c_{j-1}^{-1} \otimes I) Z_j \quad (3.7)$$

and dispersion matrix

$$c_{jj}^* \Gamma_N^*, \text{ for } j = 2, \dots, r. \quad (3.8)$$

Let us then write $\eta_1 = \phi(\xi_1)$ and for $2 \leq j \leq r$,

$$\eta_j = \phi(\xi_j) - (c'_{j-1} c_{j-1}^{-1} \otimes I)[(\phi(\xi_1))', \dots, (\phi(\xi_{j-1}))']'. \quad (3.9)$$

Then, the total hypothesis H in (3.2) may be written as $H = H_1^* \cap \dots \cap H_r^*$, where

$$H_j^*: \quad \eta_j = 0, \text{ for } j = 1, \dots, r. \quad (3.10)$$

We desire to provide a STP for H_j^* , $j = 1, \dots, r$. For this, let

$$\eta_j^* = \phi^*(\xi_j) - (c'_{j-1} c_{j-1}^{-1} \otimes I)[(\phi^*(\xi_1))', \dots, (\phi^*(\xi_{j-1}))']', \quad (3.11)$$

for $j = 2, \dots, r$ and $\eta_1^* = \phi^*(\xi_1)$, and let

$$Q_{N,j}^* = (N/c_{jj}^*)[(\eta_j^*)'(\hat{\Gamma}_N^*)^{-1}(\eta_j^*)], \quad j = 1, \dots, r. \quad (3.12)$$

By (2.32), (2.39), (3.7) and (3.8), we conclude that under H , $Q_{N,1}^*, \dots, Q_{N,r}^*$ are asymptotically independent, each having a central chi-square distribution with p degrees of freedom. Hence, there exists a Q_α^* such that

$$\lim_{N \rightarrow \infty} P\{Q_{N,s}^* \leq Q_\alpha^*, s = 1, \dots, r \mid H\} = 1 - \alpha. \quad (3.13)$$

Our proposed STP consists in testing for H_1^*, \dots, H_r^* sequentially. That is, if $Q_{N,1}^* > Q_\alpha^*$, reject H_1^* and hence H ; otherwise proceed to test for H_2^* . If $Q_{N,2}^* > Q_\alpha^*$, reject H_2^* and hence H ; otherwise proceed to H_3^* , and so on. The total hypothesis H is accepted iff $Q_{N,j}^* \leq Q_\alpha^*$ for all $j = 1, \dots, r$. By virtue of (3.7) and (3.8), we obtain the following associated simultaneous confidence intervals

$$|N^{1/2}(\eta_j^* - \eta_j)'t| \leq [Q_\alpha^* (t'(\hat{\Gamma}_N^*)t)c_{jj}^*]^{1/2} \quad j = 1, \dots, r, \quad t \neq 0, \quad (3.14)$$

with an asymptotic confidence coefficient equal to $1 - \alpha$.

Procedure III. Let $\tau' = (\tau'_1, \dots, \tau'_c)$, and let

$$\underline{a}' = (a_{11}, a_{12}, \dots, a_{1p}, \dots, a_{c1}, a_{c2}, \dots, a_{cp}) \neq 0' \quad (3.15)$$

be a pc-vector such that a_{1i}, \dots, a_{ci} are not all equal for $i = 1, \dots, p$; the totality of all possible \underline{a} satisfying the above condition is denoted by \mathcal{A} . Then, the hypothesis H in (3.2) may be written as $H = \bigcap_{\underline{a} \in \mathcal{A}} H_{\underline{a}}$, where

$$H_{\underline{a}} : \underline{a}'\tau = 0 \text{ for } \underline{a} \in \mathcal{A}. \quad (3.16)$$

We want to consider a STP for all $H_{\underline{a}}$, $\underline{a} \in \mathcal{A}$.

Let us denote by $\hat{\Delta}' = (\hat{\Delta}'_1, \dots, \hat{\Delta}'_c)$, where the $\hat{\Delta}'_k$ are defined by (2.17), and let $\underline{n} = \text{Diag}(n_1, \dots, n_c)$. Define then

$$Q_{N,o} = \hat{\Delta}'(\underline{n} \otimes (\hat{\Gamma}_{\underline{N}}^*)^{-1}) \hat{\Delta} = \sum_{k=1}^c n_k \hat{\Delta}'_k (\hat{\Gamma}_{\underline{N}}^*)^{-1} \hat{\Delta}_k. \quad (3.17)$$

By (2.32) and (2.39), under H , $Q_{N,o}$ has asymptotically chi-square distribution with $p(c-1)$ degrees of freedom. Hence,

$$\lim_{N \rightarrow \infty} P \{ Q_{N,o} \leq \chi_{p(c-1), \alpha}^2 \mid H \} = 1 - \alpha. \quad (3.18)$$

Replacing $\hat{\Delta}$ by $\hat{\Delta} - \tau$ in (3.17), and thereby eliminating H in (3.18), we obtain by the Schwartz inequality that asymptotically with a confidence coefficient $1 - \alpha$,

$$|\underline{a}'(\hat{\Delta} - \tau)| \leq \chi_{p(c-1), \alpha} [\underline{a}(\underline{n}^{-1} \otimes \hat{\Gamma}_{\underline{N}}^*) \underline{a}]^{1/2}, \forall \underline{a} \in \mathcal{A}, \quad (3.19)$$

which provides a simultaneous confidence region to all possible $\underline{a}'\tau$, $\underline{a} \in \mathcal{A}$.

The STP consists in rejecting those $H_{\underline{a}}$ ($\underline{a} \in \mathcal{A}$) for which $|\underline{a}'\hat{\Delta}| > \chi_{p(c-1), \alpha} [\underline{a}'(\underline{n}^{-1} \otimes \hat{\Gamma}_{\underline{N}}^*) \underline{a}]^{1/2}$. A similar procedure for MANOVA (but involving a loss of A.R.E. for contrasts not involving all the c samples) is due to Gabriel and Sen(1968).

Procedure III has the maximum flexibility to all $\underline{a} \in \mathcal{A}$, but, in practice, when one may be interested in a set of specified contrasts, it is usually less efficient than the two other procedures. The simplicity of the computation of Q_{α}^* over Q_{α} (derived from the asymptotic independence of $Q_{N,1}^*, \dots, Q_{N,r}^*$) has to be weighed against the arbitrariness and complications involved in the choice of the sequence of the suffixes $1, \dots, r$ and the formulation of H_1^*, \dots, H_r^* . One may also consider some other procedures along the lines of Krishnaiah(1969).

4. ARE RESULTS

One could have ignored the covariates totally, and proceeding as in Puri and Sen(1968), obtained rank order estimates of the contrasts derived solely from the $T_{k\ell}^{(i)}$, $1 \leq k < \ell \leq c$, $i = 1, \dots, p$. In this case, the asymptotic covariance matrix in (2.30) or (2.32) would have involved Γ instead of Γ^* , where

$$\Gamma = ((\gamma_{ij})) , \quad \gamma_{ij} = v_{ij,11}/B_i B_j, \quad i, j = 1, \dots, p, \quad (4.1)$$

and $v_{ij,11}$ and B_i are defined by (2.20) and (2.26). Thus, as in Puri and Sen(1968), defining the ARE as the reciprocal of the ratio of the generalized variances raised to the power p^{-1} , the ARE of the covariate adjusted rank order estimates with respect to the unadjusted ones is

$$\begin{aligned} e_1 &= \{ |\Gamma|/|\Gamma^*| \}^{1/p} = \{ |v_{11}|/|v^*| \}^{1/p} \\ &= \{ |v_{11}|/|v_{11} - v_{12}v_{22}^{-1}v_{21}| \}^{1/p}, \end{aligned} \quad (4.2)$$

where $v_{12}v_{22}^{-1}v_{21}$ is positive semi-definite. Hence, we have

$$e_1 \geq 1, \text{ where the equality sign holds when } v_{12} = 0. \quad (4.3)$$

This explains the asymptotic supremacy of the covariate adjusted estimates. It also follows similarly that the ARE is a non-decreasing function of the number of concomitant variates, i.e., additional covariates induce more information on the estimates.

For the classical normal theory estimates of contrasts based on the sample mean vectors, let $\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$ be the covariance matrix of $(Y_{\alpha}^{(k)}, X_{\alpha}^{(k)})'$, and let

$$\Sigma^* = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}. \quad (4.4)$$

Then, from the results of Sen and Puri(1970), we may conclude (on omitting the details) that the ARE of $\phi^*(\ell)$ with respect to the classical normal theory estimator of $\phi(\ell)$ is

$$e_2 = \{ |\Sigma^*|/|\Sigma| \}^{1/p}, \quad (4.5)$$

which depends on Σ_{11} , Σ_{22} , Σ_{12} as well as v_{11} , v_{22} , v_{12} and B_1, \dots, B_p . In

general, it is not possible to attach suitable bounds to e_2 . However,

$$\{|\Sigma^*|/|\Gamma|\}^{1/p} \leq e_2 \leq \{|\Sigma_{11}|/|\Gamma^*|\}^{1/p}. \quad (4.6)$$

In particular, if $\Sigma_{12} = 0$, $e_2 \geq \{|\Sigma|/|\Gamma|\}^{1/p}$, where the equality sign holds if $\Sigma_{12} = 0$. Thus, if the original variates are uncorrelated but not necessarily independent, we expect to gain in efficiency by considering the robust estimates $\phi^*(\ell)$. The same ARE results holds for the simultaneous confidence intervals in (2.40) when compared with the parallel procedure for the MANOVA model and the parametric Roy-Bose type procedure.

For the simultaneous tests in section 3, the ARE with respect to their parametric counterparts considered in detail in Krishnaiah(1969) depends on the particular set of variates or samples included in the set of contrasts under test. The overall ARE agrees with e_1 or e_2 in the respective cases, while the minimum and maximum (over all possible choice of variates and samples) ARE are respectively the minimum and the maximum characteristic roots of $\Sigma^*(\Gamma^*)^{-1}$. Since, these bounds are similar to the ones discussed in Sen and Puri (1970), we omit the details here. In passing, we may remark that on using the normal scores statistics for deriving the robust estimates in section 2, we are able to achieve asymptotic optimality when the underlying distribution is normal.

REFERENCES

- Gabriel, K.R., and Sen, P.K. (1968): Simultaneous test procedures for one-way AVOVA and MANOVA based on rank scores. Sankhya, Ser. A. 30, 303-312.
- Ghosh, M., and Sen, P.K. (1971): On a class of rank order tests for regression with partially informed stochastic predictors. Ann. Math. Statist. 42, 650-661.
- Krishnaiah, P.R. (1964): Iterated tests for the equality of mean vectors from multivariate normal populations (Abstract). Ann. Math. Statist. 35, 1844.
- Krishnaiah, P.R. (1969): Simultaneous test procedures under general MANOVA models. Multivariate Analysis-II (Ed: P.R. Krishnaiah), Academic Press, New York.
- Puri, M.L., and Sen, P.K. (1966): On a class of multivariate multisample rank order tests. Sankhya, Ser. A 28, 353-376.
- Puri, M.L., and Sen, P.K. (1968): On a class of rank order estimates of contrasts in MANOVA. Sankhya, Ser. A 30, 31-36.
- Puri, M.L., and Sen, P.K. (1969): Analysis of covariance based on general rank scores. Ann. Math. Statist. 40, 610-618.
- Puri, M.L., and Sen, P.K. (1971): Nonparametric Methods in Multivariate Analysis. John Wiley and Sons, Inc., New York.
- Quade, D. (1967): Rank analysis of covariance. Jour. Amer. Statist. Assoc. 62, 1187-1200.
- Roy, S. N. and Bose, R. C. (1953). Simultaneous confidence interval estimation. Ann. Math. Statist. 24 513-536.
- Sen, P.K., and Puri, M.L. (1970): Asymptotic theory of likelihood ratio and rank order tests in some multivariate linear models. Ann. Math. Statist. 41, 87-100.